An extension of Chapple's formula by Blaschke-like maps

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Abstract

The distance d between the circumcenter and incenter of a triangle is given by $d^2 = R(R-2r)$, where R and r are the circumradius and inradius, respectively. In particular, if circumscribed circle is the unit circle, then the distance is given by

$$d^2 = 1 - 2r.$$

This formula known as Chapple' formula is independently given by Chapple [Cha46] and Euler (1765).

Using the geometric properties of Blaschke products given by Daepp et. al [DGM02] and Frantz [Fra04], the inner circle of Chapple's theorem can be extended to an ellipse.

In this talk, we define the Blaschke-like map on the elliptic domain using conformal deformation and study the geometric properties of the maps. In particular, we give an extension of Chapple's theorem and show that each Poncele's triangle for two nested ellipses is constructed from a Blaschke-like map.

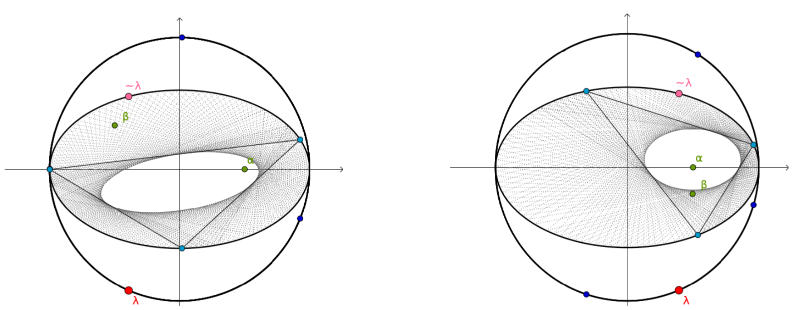

Figure 1: The envelopes indicate the interior curves of the Blashcke-like maps associated with the canonical Blashcke products with zeros 0, $\frac{1}{2}$, $-\frac{1}{2}+\frac{1}{3}i$ and $\varphi_{3/5}$ (left) and 0, $\frac{1}{2}$, $\frac{1}{2}-\frac{1}{5}i$ and $\varphi_{3/5}$ (right), where $\varphi_t(w) = \frac{1}{1+t^2}\left(t^2 w + \dfrac{1}{w}\right)$.

References

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