On F5 Algorithm for Weyl Algebras over Fields with Valuations

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K : a field with valuation
$K[\mathrm{x}]$: polynomial ring $K[x_1, \cdots, x_n]$
$D_n(K)$: Weyl algebra $K\langle x_1, \cdots, x_n, \partial_{x_1}, \cdots, \partial_{x_n}\rangle$

Definition 1

A valuation on K is a function $\mathrm{val} : K^* \to \mathbb{R}$ satisfying

$$\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b) \quad \text{and} \quad \mathrm{val}(a+b) \geq \min(\mathrm{val}(a), \mathrm{val}(b)).$$

Example.
Let $K = \mathbb{Q}$, and p be a prime number.

- For $q = p^m \frac{a}{b} \in \mathbb{Q}^*$ where p does not divide both a and b, we can define $\mathrm{val}(q) = m$.
- It is known as p-adic valuation.

Previous results by other researchers

- *Groebner bases over fields with valuations*
  (A.W.Chan and D.Maclagan: 2019)
  - $<_c$: tropical term ordering on $K[\mathrm{x}]$
    $a\mathrm{x}^\alpha <_c b\mathrm{x}^\beta$ if $\mathrm{val}(a) + u \cdot \alpha > \mathrm{val}(b) + u \cdot \beta$; using a monomial order (lex,...,etc.) as tie breaker.
  - using a 'special' division algorithm (a modified Mora's algorithm)

- *On affine tropical F5 algorithms*
  (Tristan Vaccon, T.Verron, K.Yokoyama (2021))
  - $<_v$: tropical term ordering on $K[\mathrm{x}]$
    $a\mathrm{x}^\alpha <_v b\mathrm{x}^\beta$ if $\deg(x^\alpha) < \deg(x^\beta)$; using $<_c$ as tie breaker.
  - using the tropical LUP algorithm

- *On tropical Groebner bases for rings of differential operators* $D_n(K)$
  (Ari Dwi Hartanto and K.Ohara: presented on Risa/Asir Conference 2022)

Tropical term ordering for $D_n(K)$

Definition 2 (presented on Risa/Asir Conference 2022)

$u, v \in \mathbb{R}^n$
$<_1$: a monomial ordering on $\mathcal{M} \subset D_n(K)$.

For $a\mathrm{x}^\alpha\partial^\beta, b\mathrm{x}^{\alpha'}\partial^{\beta'} \in \mathcal{T} \subset D_n(K)$, we write:

(1) $a\mathrm{x}^\alpha\partial^\beta < b\mathrm{x}^{\alpha'}\partial^{\beta'}$ if:

- (i) $|\beta| < |\beta'|$; or
- (ii) $|\beta| = |\beta'|$ and $\mathrm{val}(a) + (u,v)\cdot(\alpha,\beta) > \mathrm{val}(b) + (u,v)\cdot(\alpha',\beta')$; or
- (iii) $|\beta| = |\beta'|$ and $\mathrm{val}(a) + (u,v)\cdot(\alpha,\beta) = \mathrm{val}(b) + (u,v)\cdot(\alpha',\beta')$ and $\mathrm{x}^\alpha\partial^\beta <_1 \mathrm{x}^{\alpha'}\partial^{\beta'}$.

(2) $a\mathrm{x}^\alpha\partial^\beta = b\mathrm{x}^{\alpha'}\partial^{\beta'}$ if $\mathrm{val}(a) = \mathrm{val}(b)$ and $(\alpha,\beta) = (\alpha',\beta')$.

and call $\leq$ a tropical term order on $\mathcal{T}$.

Term ordering with valuation

We generalize Definition 2

Definition 3

$w, \omega \in \mathbb{R}^{2n}$ s.t. $w_i \geq 0$ and $\max_{1 \leq j \leq n} w_j < w_{n+i}$ for all $i = 1, \cdots, n$,

$<_1$: a monomial ordering on $\mathcal{M} \subset D_n(K)$.

For $a\mathrm{x}^\alpha \partial^\beta, b\mathrm{x}^{\alpha'} \partial^{\beta'} \in \mathcal{T} \subset D_n(K)$, we write $a\mathrm{x}^\alpha \partial^\beta < b\mathrm{x}^{\alpha'} \partial^{\beta'}$ if:

(i) $w \cdot (\alpha, \beta) < w \cdot (\alpha', \beta')$; or

(ii) $w \cdot (\alpha, \beta) = w \cdot (\alpha', \beta')$ and $-\mathrm{val}(a) + \omega \cdot (\alpha, \beta) < -\mathrm{val}(b) + \omega \cdot (\alpha', \beta')$; or

(iii) $w \cdot (\alpha, \beta) = w \cdot (\alpha', \beta')$ and $-\mathrm{val}(a) + \omega \cdot (\alpha, \beta) = -\mathrm{val}(b) + \omega \cdot (\alpha', \beta')$ and $\mathrm{x}^\alpha \partial^\beta <_1 \mathrm{x}^{\alpha'} \partial^{\beta'}$.

Moreover, we write $a\mathrm{x}^\alpha \partial^\beta = b\mathrm{x}^{\alpha'} \partial^{\beta'}$ if $\mathrm{val}(a) = \mathrm{val}(b)$ and $(\alpha, \beta) = (\alpha', \beta')$.

Remark.

If we restrict our ring to $K[\mathrm{x}] \subset D_n(K)$, we can say that

$\leq$ is more general than $\leq_c$ and $\leq_v$.

- $\leq \Leftrightarrow \leq_c$ when $w = (0, \cdots, 0, w_{n+1}, \cdots, w_{2n})$ and $(\omega_1, \cdots, \omega_n) = -1 \cdot u$.
- $\leq \Leftrightarrow \leq_v$ when $w = (1, \cdots, 1, w_{n+1}, \cdots, w_{2n})$ and $(\omega_1, \cdots, \omega_n) = -1 \cdot u$.

Division Algorithm w.r.t. $\leq$?

- adopting the division algorithm as in A.W.Chan (2019)
- the algorithm only work in homogeneous polynomials

$$D_n^{(h)}(K) := K[h]\langle x_1, \cdots, x_n, \partial_1, \cdots, \partial_n \rangle$$

  - h : the homogenization variable
  - h commutes with x_i and ∂_i
  - multiplication: $\partial_i x_i = x_i \partial_i + h^2$

Definition 4 (Term order with valuation on $D_n^{(h)}(K)$)

$w, \omega \in \mathbb{R}^{2n}$ s.t. $0 \leq w_i$ and $\max_{1 \leq j \leq n} w_j < w_{n+i}$ for all $i = 1, \cdots, n$

We write $a\mathrm{x}^\alpha \partial^\beta h^\gamma <_h b\mathrm{x}^{\alpha'} \partial^{\beta'} h^\gamma$ **if:**

(i) $\deg(\mathrm{x}^\alpha \partial^\beta) + \gamma < \deg(\mathrm{x}^{\alpha'} \partial^{\beta'}) + \gamma'$; or

(ii) $\deg(\mathrm{x}^\alpha \partial^\beta) + \gamma = \deg(\mathrm{x}^{\alpha'} \partial^{\beta'}) + \gamma'$ and $a\mathrm{x}^\alpha \partial^\beta < b\mathrm{x}^{\alpha'} \partial^{\beta'}$.

We write $a\mathrm{x}^\alpha \partial^\beta h^\gamma =_h b\mathrm{x}^{\alpha'} \partial^{\beta'} h^{\gamma'}$ **if** $\mathrm{val}(a) = \mathrm{val}(b)$ and $(\alpha, \beta, \gamma) = (\alpha', \beta', \gamma')$.

(Input/Output) Division Algorithm w.r.t. $\leq_h$

Input: A homogeneous polynomial $f \in D_n^{(h)}(K)$,
$G = \{g_1, \cdots, g_N\} \subset D_n^{(h)}(K)$ a finite set of homogeneous polynomials.

Output: $H_1, \cdots, H_N, r \in D_n^{(h)}(K)$ satisfying

$$f = \left(\sum_{i=1}^{N} H_i g_i \right) + r$$

where:

(i) $\mathrm{LT}(f) \geq_h \mathrm{LT}(H_i g_i)$ for all i

(ii) no term of r is divisible by any $\mathrm{LM}(g_i)$.

Module of syzygies

For simplicity, let us denote $R := D_n^{(h)}(K)$.

$F := \{f_1, \cdots, f_N\}$ a generating set of a left ideal $I \subset R$

We assume that they are ordered increasingly by $\leq_h$ on their leading terms.

R^N: the free R-module generated by $\mathbf{e}_1, \cdots, \mathbf{e}_N$

$\mathbb{T} := \{t\mathbf{e}_i \mid t \in \mathcal{T}, i = 1, \cdots, N\}$ the set of all module terms in R^N

Define the (surjective) R-module homomorphism:

$$\begin{array}{rccl} \nu : R^N & \longrightarrow & I \\ \mathbf{e}_i & \mapsto & f_i. \end{array}$$

$\Rightarrow$ $\mathrm{Syz}(F) := \ker(\nu)$ the module of syzygies of F

$\Rightarrow$ $R^N/\mathrm{Syz}(F) \cong I$ (by the fundamental theorem of module homomorphism)

Module Term Ordering

Definition 5

$\leq_h$: a term order on $\mathcal{T} \subset R$ with valuation.

$\prec_h$ denotes the module term order on $\mathbb{T} \subset R^N$, defined as follows:

$$ax^\alpha \partial^\beta h^\gamma \mathbf{e}_i \prec_h bx^{\alpha'} \partial^{\beta'} h^{\gamma'} \mathbf{e}_j \iff i < j \text{ or } i = j \text{ and } ax^\alpha \partial^\beta h^\gamma <_h bx^{\alpha'} \partial^{\beta'} h^{\gamma'}$$

$$ax^\alpha \partial^\beta h^\gamma \mathbf{e}_i =_h bx^{\alpha'} \partial^{\beta'} h^{\gamma'} \mathbf{e}_j \iff i = j \text{ and } ax^\alpha \partial^\beta h^\gamma =_h bx^{\alpha'} \partial^{\beta'} h^{\gamma'}$$

Natural Signatures $\mathfrak{s}_\eta$

- For $f \in I$, denote $\boldsymbol{f}$ an element of R^N s.t. $\nu(\boldsymbol{f}) = f$.
- $R^N/\mathrm{Syz}(F) \cong I$
  Let $\psi : I \longrightarrow R^N/\mathrm{Syz}(F)$ be the (unique) module isomorphism.
  $$f \mapsto \boldsymbol{f} + \mathrm{Syz}(F)$$
- For a function $\eta : R^N/\mathrm{Syz}(F) \longrightarrow R^N$ defined by $\boldsymbol{f} + \mathrm{Syz}(F) \mapsto \boldsymbol{f}$, we define a function

$$\begin{aligned} \mathfrak{s}_\eta : I\backslash\{0\} &\longrightarrow \mathbb{T}\backslash\{0\} \\ f &\mapsto \mathfrak{s}_\eta(f) := \mathrm{LT}((\eta \circ \psi)(f)). \end{aligned}$$

Property of $\mathfrak{s}_\eta$

- $\mathrm{LT}(\mathrm{Syz}(F)) \subset \mathbb{T}$ **the set of leading module terms** of $\mathrm{Syz}(F)$
- $\mathrm{NS}(\mathrm{Syz}(F)) := \mathbb{T}\backslash\mathrm{LT}(\mathrm{Syz}(F))$ **the normal set** of $\mathrm{Syz}(F)$

Lemma 6

Let $f \in I\backslash\{0\}$ and $t \in \mathcal{T}\backslash\{0\}$.
Then:

1. $\mathfrak{s}_\eta(tf) =_h \mathrm{LT}(t\ \mathfrak{s}_\eta(f)) \quad \Leftrightarrow \quad \mathrm{LT}(t\ \mathfrak{s}_\eta(f)) \in \mathrm{NS}(\mathrm{Syz}(F))$
2. $\mathfrak{s}_\eta(tf) \prec_h \mathrm{LT}(t\ \mathfrak{s}_\eta(f)) \quad \Leftrightarrow \quad \mathrm{LT}(t\ \mathfrak{s}_\eta(f)) \in \mathrm{LT}(\mathrm{Syz}(F))$

Definition 7 (top $\mathfrak{s}$-reduction)

$f, g \in I\backslash\{0\}$, and $h \in I$

We say that f (top) $\mathfrak{s}$-reduces to h by g w.r.t. $\mathfrak{s}_\eta(f)$ if there exists $t \in \mathcal{T}$ s.t.:

(i) $f - tg = h$ and $\mathrm{LT}(tg) = \mathrm{LT}(f)$,

(ii) $\mathfrak{s}_{\eta'}(tg) \prec_p \mathfrak{s}_\eta(f)$.

The polynomial g is called $\mathfrak{s}$-reductor for f

Remark.
Performing consecutively a sequence of $\mathfrak{s}$-reduction steps $f \xrightarrow{g_1} f_{(1)} \xrightarrow{g_2} f_{(2)} \xrightarrow{g_3} \cdots$ might not stop. ($<_h$ is not well-order)

Algorithm 1: Sig-Top-Reduction

Input: $f \in D_n^{(h)}(K)$ homogeneous, $\mathfrak{s}_\eta(f)$, $G = \{g_1, \cdots, g_N\} \subset D_n^{(h)}(K)$ homogeneous

Output: $H_1, \cdots, H_N, r \in D_n^{(h)}(K)$ s.t.

$$f = \left(\sum_{i=1}^{N} H_i g_i\right) + r$$

where:

(i) $\mathrm{LT}(f) \geq_h \mathrm{LT}(H_i g_i)$

(ii) no $\mathfrak{s}$-reductor in G for $\mathrm{LT}(r)$

(iii) $\mathfrak{s}_\eta(f) \succ_h \mathrm{LT}(H_i \mathfrak{s}_{\eta_i}(g_i))$.
(This implies $\mathfrak{s}_{\eta'}(r) =_h \mathfrak{s}_\eta(f)$)

1 **Initialize**
2 $T \leftarrow G$
3 $q_0 \leftarrow f, \quad j \leftarrow 0,$
$h_{1,0} \leftarrow 0, \cdots, h_{N,0} \leftarrow 0$
4 **While** $q_j \neq 0$ **do**
5 **If** there is no a sig-reductor in T for $\mathrm{LT}(q_j)$ **then**
6 **Return:** $r := q_j; \qquad H_i = h_{i,j}$
7 **else**
8 **Choose** a $\mathfrak{s}$-reductor $g \in T$ for $\mathrm{LT}(q_j)$ with $E(q_j, m_\alpha g)$ min. among all choices.
9 **Update** $T \leftarrow T \cup \{q_j\}$ if $E(q_j, m_\alpha g) > 0$
10 $q' \leftarrow q_j - c_\alpha m_\alpha g$ (canceling $\mathrm{LT}(q_j)$)
11 **If** $g \in G$ **then**
12 $q_{j+1} \leftarrow q'$
13 $\cdots$
14 **If** $g \notin G$ **then**
15 $q_{j+1} \leftarrow \frac{1}{1 - c_\alpha} q'$
16 $\cdots$
17 $j = j + 1$
18 **Return** $r := q_j; \qquad H_i = h_{i,j}$

Definition 8

We say that $f \in I$ is $\mathfrak{s}$-**irreducible** w.r.t. $\mathfrak{s}_\eta(f)$ if:

- $f = 0$; or
- there is no $g \in I$ which (top) $\mathfrak{s}$-reduces f. ($\Leftrightarrow$ there is no $\mathfrak{s}$-reductor for $\mathrm{LT}(f)$)

Definition 9

We say that an $\mathfrak{s}$-irreducible $f \in I\backslash\{0\}$ is **primitive** $\mathfrak{s}$-**irreducible** w.r.t. $\mathfrak{s}_\eta(f)$ if there is no $\mathfrak{s}$-irreducible $g \in I\backslash\{0\}$ and $t \in \mathcal{T}\backslash\{1\}$ s.t.:

(i) $\mathrm{LT}(tg) = \mathrm{LT}(f)$; and

(ii) $\mathfrak{s}_{\eta'}(tg) =_h \mathfrak{s}_\eta(f)(f)$.

Proposition 10

Let $f \in I\backslash\{0\}$. Then, $\min\limits_\eta \mathfrak{s}_\eta(f)$ exists.

We define the function

$$\begin{array}{rccc} \mathfrak{sig} : I\backslash\{0\} & \longrightarrow & \mathrm{NS}(\mathrm{Syz}(F)) \\ f & \mapsto & \min\limits_\eta(\mathfrak{s}_\eta(f)). \end{array}$$

Theorem 11

Let $f \in I$ such that f is $\mathfrak{s}$-irreducible with respect to $\sigma = \mathfrak{s}_\eta(f)$, and f is of the form $f = \nu(\sigma + \text{lower terms})$.
Then,:

(i) $f = 0 \quad \Leftrightarrow \quad \sigma \in \mathrm{LT}(\mathrm{Syz}(F))$.

(ii) $f \neq 0 \quad \Leftrightarrow \quad \sigma \in \mathrm{NS}(\mathrm{Syz}(F)) \quad \Leftrightarrow \quad f \neq 0$ *and* $\sigma = \mathfrak{s}_\eta(f) =_h \mathfrak{sig}(f)$.

Definition 12

We say that $G \subset I$ is an $\mathfrak{s}$-**Groebner basis** if
for each $\mathfrak{s}$-irreducible $f \in I \backslash \{0\}$ w.r.t. $\mathfrak{sig}(f)$, there exist $g \in G$ and $t \in \mathcal{T}$ s.t.:

(i) $\mathrm{LT}(tg) = \mathrm{LT}(f)$ and

(ii) $\mathfrak{sig}(tg) =_h \mathfrak{sig}(f)$.

Proposition 13

If G is an $\mathfrak{s}$-Groebner basis w.r.t. $\preceq_h$, then G is a Groebner basis w.r.t. $\leq_h$.

Normal Pairs

Definition 14

$g_1, g_2 \in I \backslash \{0\}$.
Let $\mathrm{spoly}(g_1, g_2) = u_1 g_1 - u_2 g_2$.

We say that (g_1, g_2) is a **normal pair** if:

(i) g_i is a primitive $\mathfrak{s}$-irreducible w.r.t. $\mathfrak{s}_{\eta_i}(g_i)$, $i = 1, 2$;

(ii) $\mathfrak{s}_{\eta_i}(u_i g_i) =_h \mathrm{LT}(u_i\ \mathfrak{s}_{\eta_i}(g_i))$ $i = 1, 2$;

(iii) $\mathfrak{s}_{\eta_1}(u_1 g_1) \neq_h \mathfrak{s}_{\eta_2}(u_2 g_2)$.

F5 criterion

Theorem 15 (conjecture)

Let G be a set of $\mathfrak{s}$-irreducible polynomials of I s.t.:

(i) *for each $i = 1, \cdots, N$ s.t. $\mathbf{e}_i \notin \mathrm{LT}(\mathrm{Syz}(F))$, there exists $g_i \in G$ s.t. $\mathfrak{s}(g_i) = \mathbf{e}_i$;*

(ii) *for any $g_1, g_2 \in G$ s.t. (g_1, g_2) is normal, there exist $g \in G$ and $t \in \mathcal{T}$ s.t. tg is $\mathfrak{s}$-irreducible and $\mathfrak{s}(tg) =_h \mathfrak{s}(\mathrm{spoly}(g_1, g_2))$.*

Then G is a $\mathfrak{s}$-Groebner basis of I.

A problem when checking condition (ii) of normal pair definition:

$$\mathfrak{s}_\eta(tf) =_h \mathrm{LT}(t\ \mathfrak{s}_\eta(f)) \qquad \Leftrightarrow \qquad \mathrm{LT}(t\ \mathfrak{s}_\eta(f)) \notin \mathrm{LT}(\mathrm{Syz}(F))$$

(we do not know $\mathrm{LT}(\mathrm{Syz}(F))$ beforehand)

Solution?

$\longrightarrow$ Introduce a new variable S, a set of known leading terms of syzygies.

$\longrightarrow$ At the beginning, $S \leftarrow \emptyset$.

$\longrightarrow$ We enlarge S when performing sig-top-reduction algorithm on $\mathrm{spoly}(g_1, g_2)$ yields 0.

In commutative case, we have principal syzygies $f_i\mathbf{e}_j - f_j\mathbf{e}_i$;
but not in non-commutative case: $D_n^{(h)}(K)$.

Algorithm 2: Main framework

Input: $F = \{f_1, \cdots, f_N\} \subset R$ homogeneous, ordered in increasing tdegree
$\leq_h$: term ordering on $\mathcal{T} \subset R$ with val.
Output: G: a Groebner basis of the left ideal I generated by F

1 $S \leftarrow \emptyset$; $G_1 \leftarrow \{(f_1, \mathbf{e}_1)\}$; $i \leftarrow 1$;
2 **for** $j = 2, \cdots, N$ **do**
3 $r \leftarrow \mathrm{DIV}(f_j, \mathrm{poly}(G_i))$
4 **if** $r \neq 0$ **then**
5 $i \leftarrow i + 1$
6 $G_i, S \leftarrow \mathrm{SigGr}((r, \mathbf{e}_i),\ G_{i-1},\ S,\ j)$
7 **Return:** $G := \mathrm{poly}(G_i)$

Algorithm 3: SigGr( )

Input: $(r, \mathbf{e}_i)$, G_{i-1}, S, j
Output: G_i: a $\mathfrak{s}$-Groebner basis of the left ideal I generated by $f_1, \cdots, f_j$
S : leading terms of syzygies

1 $[P, S] \leftarrow \mathrm{UpdateNPairs}(\emptyset,\ (r, \mathbf{e}_i),\ G_{i-1},\ S)$
2 $G_i \leftarrow G_{i-1} \cup \{(r, \mathbf{e}_i)\}$
3 **while** $P \neq \emptyset$ **then**
4 **Prune** P by S
5 **Prune** P from rewritable polynomials
6 **Pick** $(f, \sigma) \in P$ with minimal σ
7 $(f, \sigma) \leftarrow \mathfrak{s}\text{-reduce}((f, \sigma), G)$
8 **if** $f \neq 0$ **then**
9 $[P, S] \leftarrow \mathrm{UpdateNPairs}(P,\ (f, \sigma),\ G_i,\ S)$
10 $G_i \leftarrow G_i \cup \{(f, \sigma)\}$
11 **else**
12 $S \leftarrow S \cup \{\sigma\}$
13 **Return:** G_i, S

Implementation: Risa/Asir

Settings: $R = D_3^{(h)}(\mathbb{Q})$, 3-adic valuation, deglex $\prec$

$w = (0, 0, 0, 1, 1, 1)$ and $\omega = (-1, -1, -1, 1, 1, 1)$

	Buchberger Alg.			F5 Alg.*		
F	CPU time (s)	#Spoly	#ZeroReds.	CPU time (s)	#N.pairs	#ZeroReds.
$x^2 + 4x\partial_y$ $z\partial_x + 4y\partial_z$	0.2969	113	99	0.2344	71	15
$3x^2 + 2x\partial_y$ $3y\partial_x^2 + 2xy\partial_z$ $\partial_x\partial_y\partial_z + 4y^2\partial_z$	12.75	1202	1154	4.484	713	56
$3y\partial_y^2 + 2z^2\partial_z$ $2\partial_x^3\partial_z + 3x^2z\partial_y$	174.2	5047	4947	13.98	742	22

Specs.:
ThinkPad X1 Nano Gen 1
Intel® Core™ i7-1180G7 Processor (2.20 GHz up to 4.60 GHz)
16 GB LPDDR4X-4266MHz
Windows 11 Pro 64bit; Risa/Asir 20220403

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Thank You